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海外特別研究員最終報告書

独立行政法人日本学術振興会 理事長　殿

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氏　　名　鈴木康平

（氏名は必ず自署すること）

海外特別研究員としての派遣期間を終了しましたので、下記のとおり報告いたします。
なお、下記及び別紙記載の内容については相違ありません。

記

1. 用務地（派遣先国名）用務地：ピサ高等師範学校　（国名：　イタリア　）

2. 研究課題名（和文）※研究課題名は申請時のものと違わないように記載すること。
測度距離空間上の確率解析

3. 派遣期間：平成　31　年　2　月　28　日　～　令和　3　年　5　月　27　日

4. 受入機関名及び部局名
受入機関名：　ピサ高等師範学校
部局名：

5. 所期の目的の遂行状況及び成果…書式任意　**書式任意（A4判相当3ページ以上、英語で記入も可）**
（研究・調査実施状況及びその成果の発表・関係学会への参加状況等）
（注）「6.研究発表」以降については様式10－別紙1～4に記入の上、併せて提出すること。

REPORT

KOHEI SUZUKI[§]

Summary

During the stay, I conducted the following researches:

- Developing foundation of analysis and geometry on configuration spaces over singular spaces by [4, 5, 6, 7, 8] with Lorenzo Dello Schiavo (IST Austria);
- Geometric analysis on configuration spaces by [3] with Elia Brué (Princeton).

Motivation

The configuration space $\Upsilon(X)$ over the base space X is the space of all locally finite point measures on X, which has been studied in various areas of mathematics: interacting particle systems in statistical physics; infinite-dimensional metric measure geometry; representation theory of diffeomorphism groups etc.

Motivation from statistical physics:
In real-world situations, particles move in spaces with a complicated structure, e.g. molecules moving inside cells, composite materials or super-cooled media. In these environments, the motion of particles is possibly highly degenerated. In these cases, the corresponding base space X is far from being a smooth Riemannian manifold, and the degeneracy of the metric structure may consist of singularities of various types: obstacles, barriers, bottlenecks, etc.. Furthermore, particles could interact with each other in a complicated way, beyond the standard treatment within the framework of Gibbs measures. From these viewpoint, we are motived to establish the foundations of analysis and geometry of $\Upsilon(X)$ over general spaces X and invariant measures μ, aiming to include all the aforementioned singular settings.

Motivation from metric measure geometry:
In light of the developments of metric measure geometry in the last two decades, various systematic treatments of the geometry of singular spaces have been developed based on e.g., volume doubling and weak Poincaré inequalities, or the synthetic Ricci curvature bounds.

However, many important infinite-dimensional spaces lie outside the scope of these theories, being extended-metric measure spaces and displaying pathologies (distance functions are not continuous, Lipschitz functions are not necessarily measurable, metric balls are negligible sets etc). These obstacles are partially overcome by the extended-metric measure theory developed by L. Ambrosio, N. Gigli, G. Savaré [2], L. Ambrosio, M. Erbar, G. Savaré [1], and G. Savaré's lecture note (CIME Lecture note [9]).

The applicability of these abstract theory to concrete infinite-dimensional spaces remains however non-trivial. Indeed, as is indicated by measure-concentration phenomena, Lipschitz functions on infinite-dimensional spaces are 'approximately constant', and their Cheeger energy could vanish identically. In the present case for instance, cylinder functions on $\Upsilon(X)$ — the most importan class of test functions on $\Upsilon(X)$ — are typically not Lipschitz w.r.t. the L^2-transportation distance

§ Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa PI, Italy.
kohei.suzuki@sns.it.

d_2. Thus, until one can show the non-triviality of Cheeger energies, the aforementioned abstract theory does not provide any concrete information.

One of our motivations in this project is to establish solid foundations for the metric measure geometry in the case of the configuration space $\Upsilon(X)$, and to understand the corresponding statistical-physical diffusion structure in terms of infinite-dimensional metric measure theory.

Motivation from Random networks:
Every configuration $\gamma \in \Upsilon$ can be regarded as the set of vertices of a graph, with edges determined according to some assigned rule (e.g., two vertices are connected if their distance is less than some fixed constant). Then, each diffusion process on Υ describes the evolution of a random graph on X. These random graphs naturally model random networks in which many agents move randomly in the base space and are related based on the distance between their positions. From the viewpoint of graph theory, it is natural to allow X to be a general metric space, since networks can be embedded in a metric space, but not necessarily in a standard Euclidean space or a manifold.

Results

1. Foundation of analysis and geometry. In [4, 5], we provide a complete identification of the following two differential structures:

- *the analytic structure:* the Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ on $L^2(\Upsilon(X), \mu)$ lifted from the base space X by means of Dirichlet form theory
- *the geometric structure:* the extended metric measure space $(\Upsilon(X), \mathsf{d}_2, \mu)$ induced on configuration spaces by the L^2-transportation distance d_2.

We provide *Rademacher theorem* for $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ w.r.t. d_2 and various types of *Sobolev-to-Lipschitz properties*, we finally show that

- the Dirichlet form coincides with the Cheeger energy w.r.t. d_2
$$(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu)) = (\mathsf{Ch}_{\mathsf{d}_2}, W^{1,2}(\Upsilon, \mathsf{d}_2, \mu)).$$
- the intrinsic distance w.r.t. the Dirichlet form coincides with the L^2-transportation distance
$$\mathsf{d}_2 = \mathsf{d}_{\mathcal{E}^\mu}.$$

Our setting contains a wide class of singular spaces X and invariant measures μ. The class of base spaces we discuss includes RCD spaces, locally doubling metric measure spaces satisfying a local Poincaré inequality, and sub-Riemannian manifolds; as for μ our results include Campbell measures and quasi-Gibbs measures, in particular: Poisson measures, canonical Gibbs measures, as well as some determinantal/permanental point processes (sine_β, Airy_β, $\mathrm{Bessel}_{\alpha,\beta}$, Ginibre).

We have a number of applications of these fundamental results discussed in [5, 6, 7, 8], e.g.,

- (Curvature bound): $(\Upsilon, \mathsf{d}_2, \pi_\mathsf{m})$ satisfies $\mathrm{RCD}(K, \infty)$ if the base space $(X, \mathsf{d}, \mathsf{m})$ satisfies $\mathrm{RCD}(K, \infty)$. (π_m is the Poisson measure)
- (Varadhan short-time asymptotic): If (X, d, m) is $\mathrm{RCD}(K, \infty) \cap \mathsf{CAT}(0)$, then
$$-\lim_{t \to 0} 4t \log P_t(A, B) = \mathsf{d}_2^2(A, B), \quad A, B \subset \Upsilon(X) \text{ measurable},$$
where $P_t(A, B) = \int_A P_t \chi_B d\pi_m$ and P_t is the heat semigroup on $\Upsilon(X)$ with the invariant measure π_m.
- (L^∞-to-Lip_b-regularisation of the heat semigroup): If (X, d, m) is $\mathrm{RCD}(K, \infty)$, then
$$P_t F \in \mathrm{Lip}_b(\Upsilon, \mathsf{d}_2), \quad \forall F \in L^\infty(\Upsilon, \pi_m), \forall t > 0.$$

Many other applications are expected based on the canonical differential structure we constructed.

2. Geometric measure theory. According to geometric measure theory in the Euclidean spaces, perimeter measures can be expressed through the Gauß-Green formula in terms of the one-codimensional Hausdorff measure restricted on the reduced boundary. In [3], we develop an infinite-dimensional counterpart of the geometric measure theory in the configuration space. We construct the finite-codimensional Poisson measure and introduce the notion of the reduced boundary. After investigating its relation with the notion of capacities, we prove the Gauß-Green formula, through which perimeter measures can be expressed by the one-codimensional Poisson measure restricted on the reduced boundary. Furthermore, we give several equivalent characterisations of BV functions. As an application to statistical physics, we provide a generalised Itô's formula for the corresponding infinite-particle systems restricted on sets of finite perimeters.

References

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