【Grant-in-Aid for Scientific Research (S)】

Analysis on metric measure spaces by optimal transport theory and Markov processes

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Purpose and Background of the Research

●Outline of the Research

To characterize the shape of a figure, a mathematical theory based on the minimization of the total transportation cost of goods, called optimal transportation theory, has recently attracted attention and been actively studied. Before the advent of this method, research on Alexandrov spaces, in which the lower bound of the curvature of a figure is described only in the so-called metric space, a space containing the concept of distance, was the mainstream. Alexandrov spaces were applied as an important step toward solving the Poincaré Conjecture and the Geometrization Conjecture, and their importance is obvious today. However, there is a weaker formulation of the lower bound on the curvature of a figure (sectional curvature), and Alexandrov space was not a concept that corresponded to such a weak condition. The concept of "distance" alone is not sufficient to characterize the lower limit of such a weak curvature of a figure (called Rich curvature), and a "metric space" that also has the concept of "measure," which gives the volume of a figure, etc., is appropriate. Therefore, the concept of figure treated in the optimal transport theory is formulated in the framework of ″metric measure space. The Riemannian manifold, which is the classical framework of figures, is a tangent vector at each point with the concept of "magnitude and angle," which is called the "inner product of vectors," and this inner product is called the "Riemannian metric.″ Thanks to the Riemannian metric, Riemannian manifolds allow us to expand calculus, for example, the distance between two points or the volume of a figure. At present, the framework of figures characterized by optimal transport theory (called curvature dimension space) does not completely contain the entire Riemannian manifold, which is the classical framework of figures. The purpose of this research is to develop a theory that encompasses the entire Riemannian manifold, which is a classical concept of figure, and to develop analysis and geometry in this framework, using not only the optimal transport theory but also the theory of Markov processes that describe random phenomena, aiming for feedback and new applications in solving various problems.

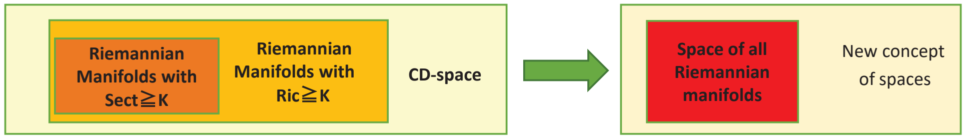

Figure 1. Image of behind of research

●Research Methodology (Research Topic)

The following six issues will be the focus of our research. (a) Generalization of the notion of rich curvature lower bounds of variable types, (b) Generalization of sub-Riemannian manifolds, which are variants of Riemannian manifolds, and their applications, (c) Research on dimensionality reduction problems using the measure concentration inequality and development of a theoretical foundation for manifold learning in machine learning theory, (d) Geometric analysis and stochastic analysis on weighted Riemannian manifolds with negative or sub-1 parameters that do not fit into the existing framework (e) Stochastic analysis of Markov processes motivated by the above research topics (f) Geometric analysis on Finsler manifolds.

●Research methodology (research plan, methods, etc.) (1) Organizing an international conference: In FY2024 (2) Domestic research meetings (3) Support for participants in the Markov Processes and its Related Fields, in the Probability Symposium and the Geometry Symposium (4) Participation in international research meetings (5) Promotion of international joint research (6) Hiring post-doctoral fellows and research assistants

・International conference
・Domestic research meetings
・Support for participants in various symposium in Japan
・Participation in International meetings
・Promotion of joint research
・Hiring post-doctoral fellows etc.

Figure 2. Diagram of research plan

Expected Research Achievements

● **Goals for each issue in the execution of this research** There is already a formulation of Riemannian curvature dimension space of variable type (called RCD(K(・),N)-space), but it imposes strong restrictions, which are removed and the Sobolev-Lipschitz property. The clarification of this property is expected to have great applications from an analytical point of view. A framework encompassing sub-Riemannian manifolds has been proposed as a quasi-curvature dimension space (called QCD space), but it does not take into account Riemannian structures, so there are many points that need to be improved. Our goal is to obtain similar results for Riemannian variable-type quasi-curvature dimensional spaces (RQCD(K(·),N)-spaces) in this framework. Some results are expected to be obtained within the research period. In addition, we are considering improvements to the measure concentration inequality, and will work closely with Shiotani and Ota, who are the project members, to obtain results that will lead to applications to manifold learning theory and other areas. On the other hand, Professors Sakurai and Xiangdong Li at the Chinese Academy of Sciences, will work on the derivation of a Liouville type theorem for V-harmonic maps from weighted Riemannian manifolds. Motivated by the aforementioned research topics, we will promote research on stochastic analysis of Markov processes and reveal useful results in terms of stochastic analysis of Markov processes. When there is no Riemannian structure (structure of inner product), the theory of stochastic analysis by Markov processes is not effective, but the analytic part is still effective to some extent. It is significant to construct a more generalized notion of curvature dimension space from nonlinear spaces such as Finsler manifolds. We will promote geometric analysis on weighted Lorenz Finsler manifolds in such a direction.

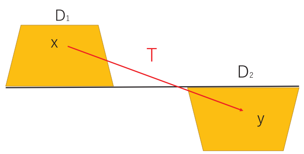

Figure 3. Image of Optimal Mass Transportation

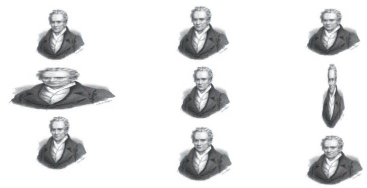

Figure 4. Diagram of optimal transport at different lower limits of curvature

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