This thesis deals with three problems on ideals in Noetherian local rings which are related to quasi-socle ideals and the Goto number of parameters, the first Hilbert coefficients of $m$-primary ideals, and almost Gorenstein rings, respectively.

In Chapter 1 we begin with examining Problem 1 below about quasi-socle ideals and the Goto number of parameters. Let $A$ be a Noetherian local ring with maximal ideal $m$ and $d = \dim A > 0$. Let $Q$ be a parameter ideal in $A$ and let $q > 0$ be an integer. We put $I = Q : m^q$ and refer to those ideals as quasi-socle ideals in $A$, or quasi-socle extensions of $Q$. Then we can ask the following problems:

Problem 1.
1. Find the conditions under which $I \subseteq \overline{Q}$, where $\overline{Q}$ stands for the integral closure of $Q$.
2. When $I \subseteq \overline{Q}$, estimate or describe the reduction number $\text{red}_Q(I) = \min \{0 \leq n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$ of $I$ with respect to $Q$ in terms of some invariants of $Q$ or $A$.
3. Clarify what kind of ring-theoretic properties of the graded rings $R(I) = \bigoplus_{n \geq 0} I^n$, $G(I) = \bigoplus_{n \geq 0} I^n$, $F(I) = \bigoplus_{n \geq 0} I^n/mI^n$ associated to the ideal $I$ enjoy.
4. Compute the Goto number $G(Q) = \{ n \in \mathbb{Z} \mid Q : m^n \subseteq \overline{Q} \}$ of $Q$.

Problem 1 is completely solved in Chapter 1 in the case where the tangent cone $G(m) = \bigoplus_{n \geq 0} m^n/m^{n+1}$ of $A$ is a Gorenstein ring and the ideal $Q$ is generated by a super regular sequence with respect to the maximal ideal $m$.

In Chapter 2 we are interested in the following Problem 2 about the first Hilbert coefficients of $m$-primary ideals. Let $A$ be a Noetherian local ring with maximal ideal $m$ and $d = \dim A > 0$. Let $l_A(M)$ denote, for an $A$-module $M$, the length of $M$. Then, for each $m$-primary ideal $I$ in $A$, we have integers $e_i(I)$ ($1 \leq i \leq d$) such that the equality

$$l_A(A/I^{n+1}) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I)$$

holds true for all integers $n \gg 0$. We call $e_i(I)$ ($1 \leq i \leq d$) the Hilbert coefficients of $A$ with respect to $I$. These integers carry a great deal of information...
about the ideal \( I \). We say that \( A \) is unmixed, if \( \dim \hat{A} / p = d \) for every \( p \in \text{Ass}(\hat{A}) \), where \( \hat{A} \) denotes the \( m \)-adic completion of \( A \).

**Problem 2.**

1. (Vasconcelos's conjecture) Assume that \( A \) is unmixed. Then \( A \) is a Cohen-Macaulay ring, once \( e_1(Q) = 0 \) for some parameter ideal \( Q \) of \( A \).
2. When is the set \( \Lambda(A) = \{ e_1(Q) \mid Q \text{ is a parameter ideal in } A \} \) finite?
3. When \( \# \Lambda(A) = 1 \)?
4. Find upper bounds and lower bounds for the set \( \{ e_1(I) \mid I \text{ is an } m - \text{ primary ideal in } A \} \).
5. Let \( I \) be an \( m \)-primary ideal in \( A \). Then is \( \{ e_1(I) \mid I \text{ is an } m - \text{ primary ideal in } A \text{ with } I = J \} \) finite?

Problem 2 will be settled in Chapter 2 and the main results are summarized into the following theorem.

**Theorem.** Let \( A \) be a Noetherian local ring with maximal ideal \( m \) and \( d = \dim A > 0 \).

1. \( A \) is a Cohen-Macaulay ring if and only if \( A \) is unmixed and \( e_1(Q) = 0 \) for some parameter ideal \( Q \) of \( A \).
2. Suppose that \( A \) is unmixed. Then \( A \) is a generalized Cohen-Macaulay ring if and only if the set \( \Lambda(A) \) is finite.
3. Suppose that \( A \) is unmixed. Then \( A \) is a Buchsbaum ring if and only if the value \( e_1(Q) \) is independent of the choice of parameter ideals \( Q \) in \( A \).
4. Suppose that \( d \geq 2 \). Then for each \( m \)-primary ideal \( I \) of \( A \), \( T_i(A) \leq e_1(I) \leq \left( \frac{e_0(I)}{2} \right) \), where \( T_i(A) \) denotes the homological torsion of \( I \). Hence the set \( \{ e_1(I) \mid I \text{ is an } m - \text{ primary ideal in } A \text{ with } I = J \} \) is finite, where \( J \) is an given \( m \)-primary ideal in \( A \).

Results (4) and (5) hold true in full generality for finitely generated \( A \)-modules. The module version for results (1), (2), and (3) are obtained either by the same technique as in the ring case or via the principle of idealization.

Chapter 3 is devoted to the following problem.

**Problem 3.** Find and develop the theory of almost Gorenstein rings.

According to Problem 3, we shall develop in Chapter 3 the theory of almost Gorenstein rings. Almost Gorenstein rings were introduced by V. Barucci and R. Froberg [BF] for analytically unramified local rings. We will give a slightly different definition
of almost Gorenstein rings, which works well also in the class of one-dimensional Cohen-Macaulay local rings which are not necessarily analytically unramified.

**Definition.** We say that $A$ is an almost Gorenstein ring, if $A$ possesses a canonical ideal $I$ such that $e_1(I) \leq r(A)$, where $r(A)$ denotes the Cohen-Macaulay type of $A$.

If $A$ is a Gorenstein ring, then we can choose any parameter ideal $Q$ of $A$ to be a canonical ideal and get $e_1(Q) = 0 < r(A) = 1$. Hence every Gorenstein local ring of dimension one is an almost Gorenstein ring. One of interesting things in this chapter is a problem which is not completely solved in [BF, Ba] can be obtained smoothly under our new definition. In the last part of this chapter, we will give a series of characterizations of almost Gorenstein rings obtained by idealization (namely, trivial extension), including the following theorem.

**Theorem.** Let $A \times m$ denote the idealization of $m$ over $A$. Then $A \times m$ is an almost Gorenstein ring if and only if $A$ is an almost Gorenstein ring.

This result enables us to construct infinitely many examples of analytically ramified almost Gorenstein rings that are not Gorenstein, which shows our modified definition enriches concrete examples of almost Gorenstein rings as well as the theory.

The results in Chapter 1 are based on the two papers [GKMP, GKPT]. The first part of Chapter 2 is already published in the paper [GhGHOPV1] and the latter sections are based on [GhGHOPV2]. Chapter 3 is based on [GMP].

**REFERENCE**


[GhGHOPV1] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. T. Phuong and W. V.
