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From Harmonic Maps to Discrete Geometry With Applications in the Materials Sciences

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Mathematics, particularly geometry, is my research field. The discipline of mathematics comprises five sub-fields or branches: algebra, geometry, fundamental analysis, mathematical analysis, and basic and applied mathematics. Algebra is the study of numbers, geometry the study of shapes, and analysis the study of differential equations whereas basic and applied mathematics have to do with establishing fundamental mathematical principles and conversely applying those principles to the various fields of science. The branches of geometry that concern me can be divided into two sub-branches. The first is "rigid geometry," which includes Riemannian geometry and differential geometry, while the second sub-branch, or "soft geometry," comprises topology and related areas. Riemannian geometry has consistently served as the backbone of my own research. However, the outcomes of my research have continued to evolve. If I am asked what my current specialty is, I am likely to reply that it is discrete geometric analysis but that I am striving to find new applications for discrete geometry through my more recent involvement with the materials sciences.

In preparing to write this essay, I reviewed the research proposals for the Grants-in-Aid funding I had received in years past and still have clear recall of my own research record and the issues that I sought to address on each occasion. For mathematicians (but not mathematicians only), the Grants-in-Aid for Scientific Research program is a foundation and a welcome source of funding that enables them to pursue research in a straightforward way that is in line with their own ideas and curiosity.Let me briefly review my research with the themes that received Grants-in-Aid in mind. Up to about the year 2000, I was engaged in the study of harmonic maps. "Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity, and affects not the pomp of superfluous causes.” As illustrated by these paraphrased quotes from Isaac Newton, the natural world under a variety of conditions selects shapes that require minimum energy. The world around us is filled with countless shapes that possess “harmony” with rich symmetries. Why do shapes of minimum energy achieve the most symmetry? Going further, how many shapes of minimum energy exist under identical conditions? Are they stable or unstable? If the energy is focused in a single point, can they explode? What kinds of shapes do towers of such focused energy points form? Finding mathematical solutions to questions of this nature is one of the challenges of the study of harmonic maps. To give an example, one of my research projects (supported by a Grant-in-Aid for Scientific Research [C]) had to do with the theme, “Chowa shazo no baburu gensho to compakuto-ka riron” (“Bubbling-off Phenomena of Harmonic Maps and Their Compactification”). In the year 2000, I was engaged in a study (supported by a Grant-in-Aid for Scientific Research [C]) on the subject, “Gurafu no chowa shazo to risan-gun no hyogen” (Harmonic maps of graphs and representations of discrete groups). This theme incorporated two key words: “harmonic maps,” a subject on which I had focused my research to that point in time, and “discrete,” an idea that became a central focus of research I pursued from that period onward. Once I realized that the knowledge I had amassed to that point on harmonic maps could be applied to the study of discrete phenomena, my research accordingly took off. I consider this to have been one of the breakthrough points in my career. Afterward, my interest shifted to discrete geometry and discrete geometric analysis, as the titles of subsequent research projects indicated. These included, for example, a 2002 project (with a Grant-in-Aid for Scientific Research [C]) on the theme, “Kessho koshi no hyojun-teki jitsugen to jiba-tsuki suii sayo-so no supekutoru kaiseki” (“The Standard Realization of Crystal Lattices and Spectra of Magnetic Transition Operators”); a 2004 project (with a Grant-in Aid for Scientific Research [B]) on the theme, “Risan-gun no sayo suru mugen gurafu no supekutoru kaiseki to guromofu-hausudorufu shusoku” (“Spectral Analysis of Infinite Graphs with Discrete Group Actions”); and a 2008 project (with a Grant-in-Aid for Scientific Research [A]) on the theme, “Randamu-sei o toshite miru risan kukan no kikagaku” (“Study of Geometry of a Discrete Space through Randomness”).

I would like to briefly explain the key word, “discrete.” As a concept, “discrete” is the antithesis of “continuous.” Here, I will forgo providing a formal definition and ask that readers simply understand “discrete” as a term conveying a disconnected condition whereas “continuous” refers to the condition of being connected. To give some examples from the material world, the atoms or molecules composing a substance are arranged and move in “discrete” patterns, but on a macroscopic scale the materials they compose appear to be “continuous” in form and motion. As symbolized by the concepts of differential and integral calculus, up through the 20th century, the discipline of mathematics was generally concerned with continuous or smooth spaces. As such, it sought to define an array of phenomena through the use of differential equations and developed various tools for that purpose. Around the end of the 20th century, a tide of interest in the analysis of “discrete” phenomena began to form and gain momentum throughout the field of mathematics, and assorted mathematical ideas and concepts were introduced for that purpose. Mikhael L. Gromov has been a huge influence in the field of geometry. He is the French-Russian mathematician who won the Kyoto Prize in 2002 for “the introduction of a metric structure for families of various geometrical objects and other contributions that have brought dramatic developments in geometry and many other fields of mathematics,” and the Abel Prize in 2009 for his “revolutionary contributions to geometry.” Although differential equations have long served as a crucial mathematical tool, in the field of discrete mathematics, they are taboo. Clearing this obstacle has been a challenge. My own work in this area has involved utilizing probability theory as an alternative to differential equations, pursuing the study of probability theory from a geometric perspective, and dynamically integrating into discrete geometric analysis the formation of geometric structures hidden in probability spaces . These approaches have allowed my research to evolve and focus on the study of condensed matter physics and ultimately themes in materials science. For example, in 2011 I embarked on a project with the theme “Bussei-butsuri ni hassuru hikakan-kikagaku moderu no teian” (Proposal for a non-commutative geometric model of condensed matter physics) (with a Grant-in-Aid for Challenging Exploratory Research), and in 2012 “Ryoshi supin-kei no risan-kika-kaiseki-gaku” (Discrete geometric analysis of quantum spin systems) (backed by a Grant-in-Aid for Scientific Research [A]).

The point I wanted to illustrate is that my interest in research has evolved with my curiosity and I have applied to grants with specific aims in mind, but its focus and values have not changed. I am deeply grateful for the freedom of scope and support that grants-in-aid have made possible.